$\exp(x) = $ inverse of $\ln(x)$

Last day, we saw that the function $f(x) = \ln x$ is one-to-one, with domain $(0, \infty)$ and range $(-\infty, \infty)$. We can conclude that $f(x)$ has an inverse function which we call the natural exponential function and denote (temorarily) by $f^{-1}(x) = \exp(x)$, The definition of inverse functions gives us the following:

$$y = f^{-1}(x) \text{ if and only if } x = f(y)$$

$$\boxed{y = \exp(x) \text{ if and only if } x = \ln(y)}$$

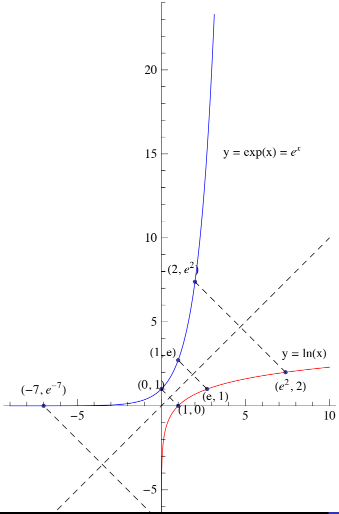
The cancellation laws give us:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

$$\boxed{\exp(\ln x) = x \quad \text{and} \quad \ln(\exp(x)) = x}.$$

Graph of exp(x)

We can draw the graph of $y = exp(x)$ by reflecting the graph of $y = ln(x)$ in the line $y = x$.



have that the graph $y = \exp(x)$ is one-to-one and continuous with domain $(-\infty, \infty)$ and range $(0, \infty)$. Note that $\exp(x) > 0$ for all values of x. We see that

$\exp(0) = 1$ since $\ln 1 = 0$
$\exp(1) = e$ since $\ln e = 1$,
$\exp(2) = e^2$ since $\ln(e^2) = 2$,
$\exp(-7) = e^{-7}$ since $\ln(e^{-7}) = -7$.

In fact for any rational number r, we have

$\exp(r) = e^r$ since $\ln(e^r) = r \ln e = r$,

by the laws of Logarithms.

Definition of e^x.

Definition When x is rational or irrational, we define e^x to be $\exp(x)$.

$$\boxed{e^x = \exp(x)}$$

Note: This agrees with definitions of e^x given elsewhere (as limits), since the definition is the same when x is a rational number and the exponential function is continuous.

Restating the above properties given above in light of this new interpretation of the exponential function, we get:
When $f(x) = \ln(x)$, $f^{-1}(x) = e^x$ and

$$\boxed{e^x = y \quad \text{if and only if} \quad \ln y = x}$$

$$\boxed{e^{\ln x} = x \quad \text{and} \quad \ln e^x = x}$$

Solving Equations

We can use the formula below to solve equations involving logarithms and exponentials.

$$\boxed{e^{\ln x} = x \quad \text{and} \quad \ln e^x = x}$$

Example Solve for x if $\ln(x+1) = 5$

- *Applying the exponential function to both sides of the equation* $\ln(x+1) = 5$, *we get*

$$e^{\ln(x+1)} = e^5$$

- *Using the fact that* $e^{\ln u} = u$, *(with* $u = x + 1$ *), we get*

$$x + 1 = e^5, \quad \text{or} \quad \boxed{x = e^5 - 1}.$$

Example Solve for x if $e^{x-4} = 10$

- *Applying the natural logarithm function to both sides of the equation* $e^{x-4} = 10$, *we get*

$$\ln(e^{x-4}) = \ln(10)$$

- *Using the fact that* $\ln(e^u) = u$, *(with* $u = x - 4$*) , we get*

$$x - 4 = \ln(10), \quad \text{or} \quad \boxed{x = \ln(10) + 4.}$$

Limits

From the graph we see that

$$\lim_{x\to-\infty} e^x = 0, \qquad \lim_{x\to\infty} e^x = \infty.$$

Example Find the limit $\lim_{x\to\infty} \frac{e^x}{10e^x - 1}$.

- *As it stands, this limit has an indeterminate form since both numerator and denominator approach infinity as $x \to \infty$*
- *We modify a trick from Calculus 1 and divide (both Numertor and denominator) by the highest power of e^x in the denominator.*

$$\lim_{x\to\infty} \frac{e^x}{10e^x - 1} = \lim_{x\to\infty} \frac{e^x/e^x}{(10e^x - 1)/e^x}$$

- $$= \lim_{x\to\infty} \frac{1}{10 - (1/e^x)} = \frac{1}{10}$$

Rules of exponentials

The following rules of exponents follow from the rules of logarithms:

$$e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}.$$

Proof see notes for details

Example Simplify $\frac{e^{x^2} e^{2x+1}}{(e^x)^2}$.

- $$\frac{e^{x^2} e^{2x+1}}{(e^x)^2} = \frac{e^{x^2+2x+1}}{e^{2x}}$$
- $$= e^{x^2+2x+1-2x} = e^{x^2+1}$$

Derivatives

$$\boxed{\frac{d}{dx}e^x = e^x} \qquad \boxed{\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}}$$

Proof We use logarithmic differentiation. If $y = e^x$, we have $\ln y = x$ and differentiating, we get $\frac{1}{y}\frac{dy}{dx} = 1$ or $\frac{dy}{dx} = y = e^x$. The derivative on the right follows from the chain rule.

Example Find $\frac{d}{dx}e^{\sin^2 x}$

- *Using the chain rule, we get*

$$\frac{d}{dx}e^{\sin^2 x} = e^{\sin^2 x} \cdot \frac{d}{dx}\sin^2 x$$

- $$= e^{\sin^2 x}2(\sin x)(\cos x) = 2(\sin x)(\cos x)e^{\sin^2 x}$$

Derivatives

$$\boxed{\frac{d}{dx}e^x = e^x} \qquad \boxed{\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}}$$

Example Find $\frac{d}{dx}\sin^2(e^{x^2})$

- *Using the chain rule, we get*

$$\frac{d}{dx}\sin^2(e^{x^2}) = 2\sin(e^{x^2}) \cdot \frac{d}{dx}\sin(e^{x^2})$$

-

$$= 2\sin(e^{x^2})\cos(e^{x^2}) \cdot \frac{d}{dx}e^{x^2}$$

-

$$= 2\sin(e^{x^2})\cos(e^{x^2})e^{x^2} \cdot \frac{d}{dx}x^2 = 4xe^{x^2}\sin(e^{x^2})\cos(e^{x^2})$$

Integrals

$$\int e^x dx = e^x + C \qquad \int g'(x)e^{g(x)}dx = e^{g(x)} + C$$

Example Find $\int xe^{x^2+1}dx$.

- *Using substitution, we let* $u = x^2 + 1$.

$$du = 2x\ dx, \qquad \frac{du}{2} = x\ dx$$

- $$\int xe^{x^2+1}dx = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$$

- *Switching back to* x*, we get*

$$= \frac{1}{2}e^{x^2+1} + C$$

Summary of formulas

$\boxed{\ln(x)}$

$$\ln(ab) = \ln a + \ln b, \quad \ln(\frac{a}{b}) = \ln a - \ln b$$

$$\ln a^x = x \ln a$$

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0} \ln x = -\infty$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, \quad \frac{d}{dx} \ln|g(x)| = \frac{g'(x)}{g(x)}$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C.$$

$\boxed{e^x}$

$$\ln e^x = x \quad \text{and} \quad e^{\ln(x)} = x$$

$$e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}.$$

$$\lim_{x\to\infty} e^x = \infty, \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0$$

$$\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} e^{g(x)} = g'(x) e^{g(x)}$$

$$\int e^x dx = e^x + C$$

$$\int g'(x) e^{g(x)} dx = e^{g(x)} + C$$

Summary of methods

Logarithmic Differentiation
Solving equations
(Finding formulas for inverse functions)
Finding slopes of inverse functions (using formula from lecture 1).
Calculating Limits
Calculating Derivatives
Calculating Integrals (including definite integrals)